12.714 Computational Data Analysis

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Today's class

- Linear Time-Invariant (LTI) Filters
  - Basic theory of analog filters
  - Basic theory of digital filters
  - Convolution as an LTI filter
  - Spectral Density Function determination
  - Interpretation of spectrum via band-pass filtering
  - Least-squares filter design
- Aim is to formalize the relation between input and output spectra.
- Also provides as way of studying stochastic processes: With some mild conditions, an input stationary process will generate an output stationary process.

Basic Theory: Analog

- A *analog filter* can be defined as a mapping between an input function and an output function: Symbolically L{x(.)}=y(.). In engineering a continuous parameter filter is called an analog filter; in mathematics called an transformation or operator.
- Some notation:
  - For a real valued constant τ and a function x(.), then x(.;τ) denotes the function whose value at time t is given by x(t+τ),
  - A filter defined by L{x(.)}=x(.;τ) is a shift filter (e.g., if x(t)=sin(t); then x(t;π/2)=cos(t)

Linear time-invariant (LTI) analog filter

- An anolog filter is linear time invariant if it has these properites:
  - Scale preservation: L{αx(.)} = αL{x(.)} where α is constant
  - Superposition: L{x(.)+y(.)}=L{x(.)}+L{y(.)}
  - Time invariance: if L{x(.)}=y(.), then L{x(.;τ)}=y(.;τ) i.e., if two inputs to the filter are the same except for a time shift; then the outputs are the same except for the same time shift.
- The first two properties express linearity and therefore

$$\boxed{L\left\{\sum_{j=1}^{N}\alpha_j x_j(.)\right\} = \sum_{j=1}^{N}\alpha_j L\left\{x_j(.)\right\}}$$

- With suitable conditions the above summation can be extended to infinity

Relationship between input and output of an LTI filter

- Use as input to the filter a complex exponential and let y(.) denote the output

$$y_f(.) = L\{\varepsilon_f(.)\} \quad \varepsilon_f(t) \equiv e^{i2\pi ft} \quad -\infty < t < \infty$$

- Using the LTI properties (3) and (1) we have

$$y_f(.;\tau) = L\{\varepsilon_f(.;\tau) = L\{e^{i2\pi f\tau}\varepsilon_f(.)\} = e^{i2\pi f\tau} y_f(\tau)$$

which implies that

$$y_f(t;\tau) = y_f(t+\tau) = e^{i2\pi f\tau} y_f(\tau) \text{ for all t and } \tau$$

- Since the above is valid for any value of t, the above implies

$$y_f(t) = e^{i2\pi ft} y_f(0) \text{ for all } t \text{ i.e., } y_f(.) = \ y_f(.)\varepsilon_f(.)$$

$$\text{Denote } G(f) = y_f(0) \text{ and we have } \mathrm{L}\{\varepsilon_f(.)\} = G(f)\varepsilon_f(.)$$

LTI relations

- We have *L{ε_f(.)}=G(f)ε_f(.)*. The complex exponentials are the *eigenfunctions* for the LTI filter and each *G(f)* is called an *associated eigenvalue*.
- Importance: If the input to an LTI filter is a complex exponential, then the output is same complex exponential multiplied by *G(f)*.
- We have

$$x(t) = \sum_f \alpha_f e^{-i2\pi ft} \text{ then } y(.) = L\{x(.)\} \quad y(t) = \sum_f \alpha_f G(f) e^{-i2\pi ft}$$

When the spectral density function (sdf) exists for x $\mathrm{S}_\mathrm{X}(f)$

$$S_Y(f) = |G(f)|^2 S_X(f)$$

Transfer function G(.)

- The G(.) is called the transfer function or frequency response function. It relates the input and output spectra and is independent of time. No power is transferred between frequencies.
- In general G(f) is complex and can be written as

$$G(f) = |G(f)|e^{i\theta(f)}$$

where *|G(f)|* is called the *gain function* and the *$\theta(f)$* is the *phase function*.

- The negative is *$\theta(f)$* called the phase shift function and the quantity $-\dfrac{1}{2\pi}\dfrac{d\theta(f)}{df}$ is called the group delay.

LTI digital filters

- The relationship between sequences (discrete rather can continuous functions) are called *digital filters*.
- For LTI filter the same properties of scale preservation, superposition, and time invariance apply.
- The *LTI digital filtering theorem* is: If *$\{X_t\}$* is a discrete stationary process with zero mean and integrated spectrum *$S^{(I)}_X(.)$* and *L* is a LTI filter with transfer function *G(.)* such that

$$\int_{-1/2}^{1/2} |G(f)|^2 dS_X^{(I)}(f) < \infty \text{ then } \{\mathrm{Y_t}\} = L\{\{X_t\}\}$$
$$dS_Y^{(I)}(f) = |G(f)|^2 dS_X^{(I)}(f)$$

- $\{Y_t\}$ is a discrete stationary process

Convolution as an LTI filter

- Consider an LTI analog filter of the following form

$$L\{X(t)\} = \int_{-\infty}^{\infty} g(u)X(t-u)du \equiv Y(t)$$

- The function *g(.)* is called the *impulse response function* because if *X(.)* is a dirac delta function the output of the convolution is *g(t)* i.e., *L{d(t)}=g(t)*
- To find the transfer function we use

$$L\{e^{i2\pi ft}\} = \int_{-\infty}^{\infty} g(u)e^{i2\pi f(t-u)}du = e^{i2\pi ft}\int_{-\infty}^{\infty} g(u)e^{-i2\pi fu}du$$

$$G(f) = \int_{-\infty}^{\infty} g(u)e^{-i2\pi fu}du \text{ This is just the Fourier Transform}$$

Convolution for digital LTI filter

- For discrete processes the same relationships apply

$$L\{X_t\} = \sum_{u=-\infty}^{\infty} g_u X_{t-u} \equiv Y_t$$

$$G(f) \equiv \sum_{u=-\infty}^{\infty} g_u e^{-i2\pi u} \text{ for } |f| \le 1/2$$

- The stationary process sdf are related by

$$S_Y(f) = |G(f)|^2 S_X(f) \text{ for } |f| \le 1/2$$

Determination of SDF by LTI filtering

- The theory developed so far allows us to determine spectral density functions for discrete stationary processes.
- Moving average process

$$X_t = \varepsilon_t - \theta_{1,q}\varepsilon_{t-1} - \theta_{2,q}\varepsilon_{t-2} - \cdots - \theta_{q,q}\varepsilon_{t-q}$$

$$\text{Define } L\{y_t\} = y_t - \theta_{1,q}y_{t-1} - \theta_{2,q}y_{t-2} - \cdots - \theta_{q,q}y_{t-q} \quad L\{\varepsilon_t\} = X_t$$

$$L\{e^{i2\pi ft}\} = e^{i2\pi ft} - \theta_{1,q}e^{i2\pi f(t-1)} - \theta_{2,q}e^{i2\pi f(t-2)} - \cdots - \theta_{q,q}e^{i2\pi f(t-q)}$$

$$= e^{i2\pi ft}(1 - \theta_{1,q}e^{-i2\pi f} - \cdots - \theta_{q,q}e^{-i2\pi fq})$$

$$G(f) = 1 - \theta_{1,q}e^{-i2\pi f} - \theta_{2,q}e^{-i2\pi f2} - \cdots - \theta_{q,q}e^{-i2\pi fq}$$

$$S_X(f) = |G(f)|^2 S_\varepsilon(f) = \sigma_\varepsilon^2\left|1 - \theta_{1,q}e^{-i2\pi f} - \cdots - \theta_{q,q}e^{-i2\pi fq}\right|^2$$

SDF of AR process

- The same process can be used for an AR process

$$X_t = \phi_{1,p}X_{t-1} + \cdots + \phi_{p,p}X_{t-p} + \varepsilon_t$$

$$\text{Define } L\{y_t\} = y_t - \phi_{1,p}y_{t-1} - \cdots - \phi_{p,p}y_{t-p} \quad \text{then } L\{X_t\} = \varepsilon_t$$

$$G(f) = 1 - \phi_{1,p}e^{-i2\pi f} - \cdots - \phi_{p,p}e^{-i2\pi fp}$$

$$S_\varepsilon(f) = |G(f)|^2 S_X(f)$$

$$S_X(f) = \frac{\sigma_\varepsilon^2}{\left|1 - \phi_{1,p}e^{-i2\pi f} - \cdots - \phi_{p,p}e^{-i2\pi fp}\right|^2}$$

- An AR process is stationary if the denominator above never goes to zero.

SDF of ARMA process

- The previous two cases can be merged for an ARMA process

$$X_t = \phi_{1,p}X_{t-1} + \cdots + \phi_{p,p}X_{t-p} + \varepsilon_t - \theta_{1,q}\varepsilon_t - \cdots - \theta_{q,q}\varepsilon_{t-q}$$

$$S_X(f) = \sigma_\varepsilon^2 \frac{\left|1 - \theta_{1,q}e^{-i2\pi f} - \cdots - \theta_{q,q}e^{-i2\pi fq}\right|^2}{\left|1 - \phi_{1,p}e^{-i2\pi f} - \cdots - \phi_{p,p}e^{-i2\pi fp}\right|^2}$$

Filter terminology

- Field is very large (see references p 169,PW)
- Types of filters:
  - *Cascaded:* An arrangement of n-filters where the output of one is the input to the next.
    - If all the filters are LTI filters, then an input stationary process will generate and output stationary process.
    - The total integrated spectrum will be given by

$$dS_{n+1}^{(I)}(f) = |G_n(f)|^2 |G_{n-1}(f)|^2 \cdots |G_1(f)|^2 dS_1^{(I)}(f)$$

    - Note for LTI filters, the output does not depend on the order of the filters.
  - *Ideal low-pass filter*: Transfer function $|G(f)|^2 = \begin{cases} 1, & \text{if } |f| \leq f_0 \\ 0, & \text{otherwise} \end{cases}$

Filter terminology

- Filter types:
  - *Ideal high-pass filter*: $\boxed{|G(f)|^2 = \begin{cases} 1, & \text{if } |f| \geq f_0 \\ 0, & \text{otherwise} \end{cases}}$
  - *Ideal band-bass filter*: In the band $[f_1, f_2]$

$$\boxed{|G(f)|^2 = \begin{cases} 1, & \text{if } 0 < f_1 \leq |f| \leq f_2 \\ 0, & \text{otherwise} \end{cases}}$$

- A notch filter is one that removes a specific frequency range (e.g., to remove diurnal effects)
- None of the ideal filters can be implemented with finite length sequences.

Filter Terminology

- Given an LTI filter of the form:

$$\boxed{L\{X_t\} = \sum_{u=-\infty}^{\infty} g_u X_{t-u}}$$

- The filter is said to be *causal* if g_u=0 for all u<0
- If g_u is zero outside a finite range of u, the filter is called a *finite impulse response (FIR)* filter.
- If this is not the case then it is an *infinite impulse response (IIR)* filter.

Interpretation of Spectrum

- Ideal band pass filters allow a physical interpretation of the integrated spectrum.
- Given an ideal band pass filter between f and f+df, then the passing on stationary process through will yield signal in narrow bandwidth. The variance of this narrow bandwidth signal will be

$$\sigma_X^2 = \int_{-\infty}^{\infty} dS_X^{(I)}(f) = \int_{-\infty}^{\infty} S_X(f)df$$

$$\left|G(f)\right|^2 = 1 \text{ if } f' \leq |\mathrm{f}| \leq f' + df'; \text{ else } = 0$$

$$S_Y(f) = \left|G(f)\right|^2 S_X(f) \text{ and hence } \sigma_Y^2 = 2S_X(f')df'$$

Example of LTI digital filtering

- Consider an acausal LTI digital filter:

$$g_u^{(1)} = \begin{cases} 1/2, & u = 0 \\ 1/4, & u = \pm 1 \\ 0, \text{ otherwise} \end{cases}$$

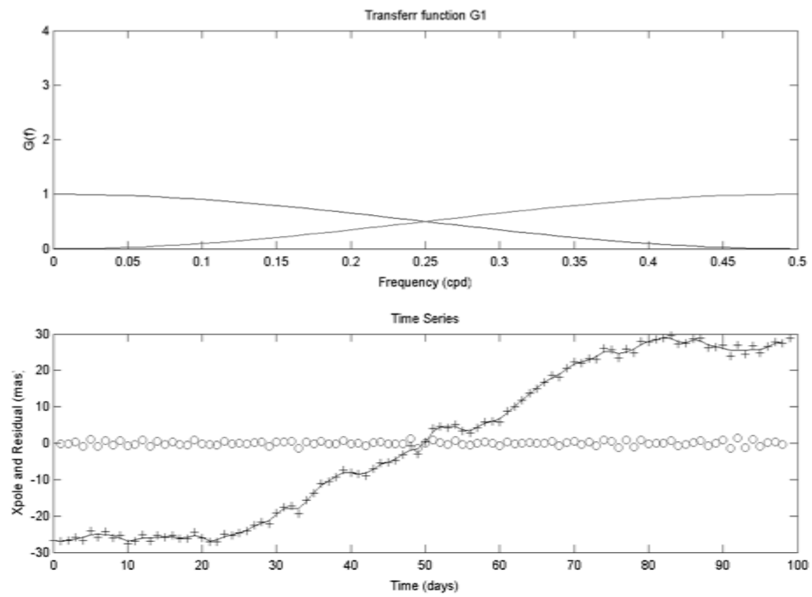
$$G^{(1)}(f) = g_{-1}^{(1)} e^{i2\pi f} + g_o^{(1)} + g_{+1}^{(1)} e^{-i2\pi f} = \cos^2(\pi f) \quad |f| \leq 1/2$$

The residual is defined by filtered signal - original signal

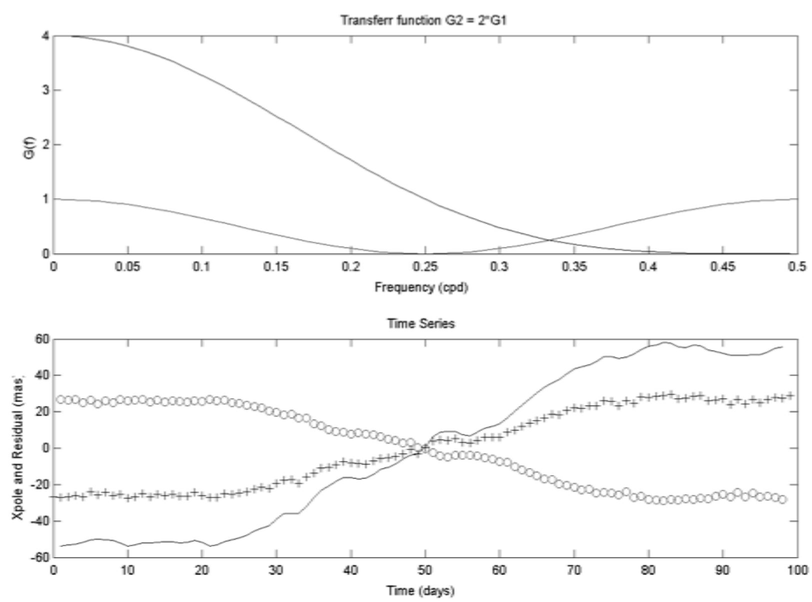
$$H^{(1)}(f) = -g_{-1}^{(1)} e^{i2\pi f} + (1 - g_o^{(1)}) - g_{+1}^{(1)} e^{-i2\pi f} = \sin^2(\pi f) \quad |f| \leq 1/2$$

- The next two figures show transfer function for above case and a filter $g^{(2)} = 2*g^{(1)}$

$g^{(1)}$ Transfer function and filter



$g^{(2)}=2*g^{(1)}$ transfer function and filter



Conditions on Transfer function

- As we just seen; an LTI filter does not necessarily behave as a smoothing filter
- If we want a filter that traces the trend of a given time series, the LTI filer must be appropriately normalized.
- Filter can be normalized by requiring that if the input function is locally smooth, the filtered value should match the input.

$$\sum_{u=K}^{K} g_u x_{t-u} = x_t \text{ when } x_t = \alpha + \beta t$$

if g_u is symmetric ($g_{-u} = g_u$) then above satisfied if $\sum_{u=K}^{K} g_u = 1$

Normalized transfer functions

- If the impulse response is symmetric and sums to 1 then, the transfer function is given by

$$G(f) = g_0 + 2\sum_{u=1}^{K} g_k \cos(2\pi f u) \quad G(0) = 1$$

The residual transfer function is $H(f) = 1 - G(f)$

- Hence to define as low pass filter whose residuals are a high pass filter, the sum of the coefficients should be one, and the filter should be symmetric. The latter is not required.

Least squares filter design

- The ideal low pass filter and its impulse response are

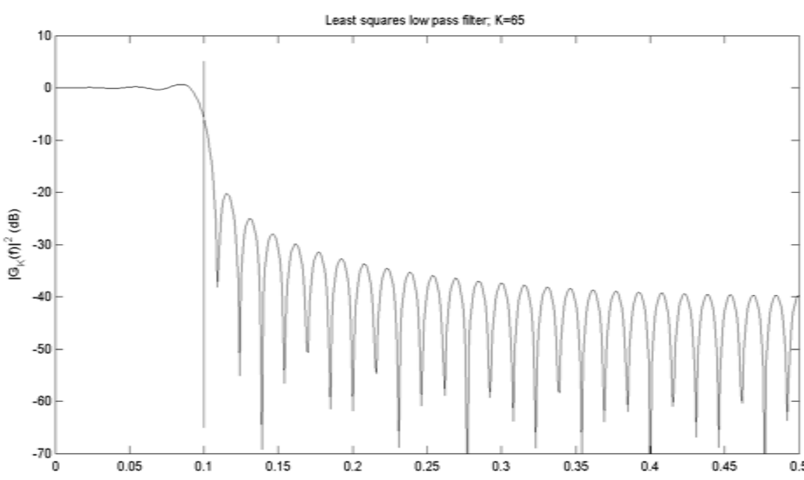
$$G_I(f) \equiv \begin{cases} 1 & \text{if } |f| \le W \\ 0, W < |f| \le 1/2 \end{cases}$$

$$g_{u,I} \equiv \int_{-1/2}^{1/2} G_I(f) e^{i2\pi f u} df = \begin{cases} 2W, & u = 0 \\ \dfrac{\sin(2\pi W u)}{\pi u}. & u \ne 0 \end{cases}$$

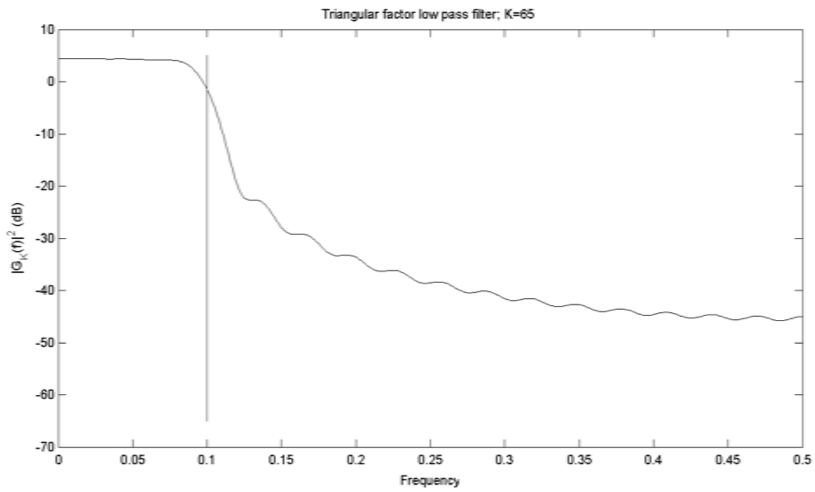
- As we have seen before the least squares fit with a

$$g_{u,K} \equiv \begin{cases} g_{u,I} & \text{if } |u| \le \lfloor K/2 \rfloor \\ 0, & \text{otherwise} \end{cases} \quad G_K(f) \equiv \sum_{u=-\lfloor K/2 \rfloor}^{u=\lfloor K/2 \rfloor} g_{u,K} e^{-i2\pi f u}$$

LSQ low pass filter K=65



Alternative low pass filter with triangular convergence factor



Low pass filters

- The previous figure was generated using:

$$g_{u,K}^{(c)} = c_u g_{u,K} \quad \text{with} \quad c_u = \begin{cases} \gamma\left(1 - \frac{2|u|}{K+1}\right), & |u| \le \lfloor K/2 \rfloor \\ 0. & \text{otherwise} \end{cases}$$

- The factor γ is included to ensure sum g_u =*1*
- Other variations on the filter design include fitting to the transfer function:

$$G_\delta(f) = \begin{cases} 1. & |f| \le W - \delta \\ \frac{1}{2}\left[1 + \cos(\pi \frac{|f| - W + \delta}{2\delta}\right], & W - \delta < |f| \le W + \delta \\ 0, & W + \delta < |f| \le 1/2 \end{cases}$$

Summary of today's class

- Linear Time-Invariant (LTI) Filters
  - Basic theory of analog filters: Continuous
  - Basic theory of digital filters: Discrete in time
  - Convolution as an LTI filter
  - SDF determination
  - Interpretation of spectrum via band-pass filtering
  - Least-squares filter design: Low pass filters
- Text book also discusses using dpss functions as low pass filters.