12.714 Computational Data Analysis

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Today's class

- Concentration Problem:
  - Signals that are near time and band limited (Chapter 3, PW) (Subject appears again when linear time-invariant filters and multi-taper spectral estimation are covered)
  - Generation of Slepian functions - Discrete Prolate Spherical Sequences (DPSS)
- Fourier theory for discrete time and discrete frequency
  - Fast Fourier Transform (FFT)

Concentration Problem: Discrete

- Concepts of the width of functions has been previously covered. We also showed that only a null function can be both band and time limited. Question: What types of functions have small widths in the both time and frequency domains?
- Original work done by Slepian (1978)
- We will define a measure of concentration such that a large value will represent functions that concentrate their power in either limited time or frequency range. The question then is for functions that are band or time limited, what function generate the most concentration in the other domain.

Concentration

- Given a real or complex sequence $\{g_t\}$ with $\Delta t=1$ (Nyquist frequency 1/2), we have

$$G_p(f) = \sum_{-\infty}^{\infty} g_t e^{-i2\pi ft} \quad \text{Parseval's Theorem } \sum_{-\infty}^{\infty} |g_t|^2 = \int_{-1/2}^{1/2} |G_p(f)|^2 df$$

$$\text{Define } \alpha^2(N) = \sum_{t=0}^{N} |g_t|^2 \Big/ \sum_{-\infty}^{\infty} |g_t|^2 \text{ and } \beta^2(W) = \int_{-W}^{W} |G_p(f)|^2 df \Big/ \int_{-1/2}^{1/2} |G_p(f)|^2 df$$

- The quantities α^2 and β^2 measure the amount of power in either a finite amount of time or bandwidth.
- The concentration problems are: How large can α^2 be for a band limited signal ($|f| \le W < 1/2$) and how large can β^2 be for a time limited (index limited for discrete case)

Concentration solution

- The derivation of the solution to the concentration problem is from Slepian (1978) and is outlined pp 101-105 of PW
- Basic solution for the discrete time case is an eigenvalue problem:

$$\int_{-W}^{W} \sum_{t=0}^{N-1} e^{i2\pi f[t-(N-1)/2]} e^{-i2\pi f'[t-(N-1)/2]} U(f) df = \lambda U(f)$$

- With U(f) as the eigenfunctions and λ the eigenvalues
- The eigenfunctions are orthonormal over [-1/2 1/2] and orthogonal over [-W W]. There are N eigenvalues.
- The first 2WN eigenvalues are close to the 1 and the remaining ones are near zero

Concentration problem

- The band-limited sequence that solves the first concentration problem

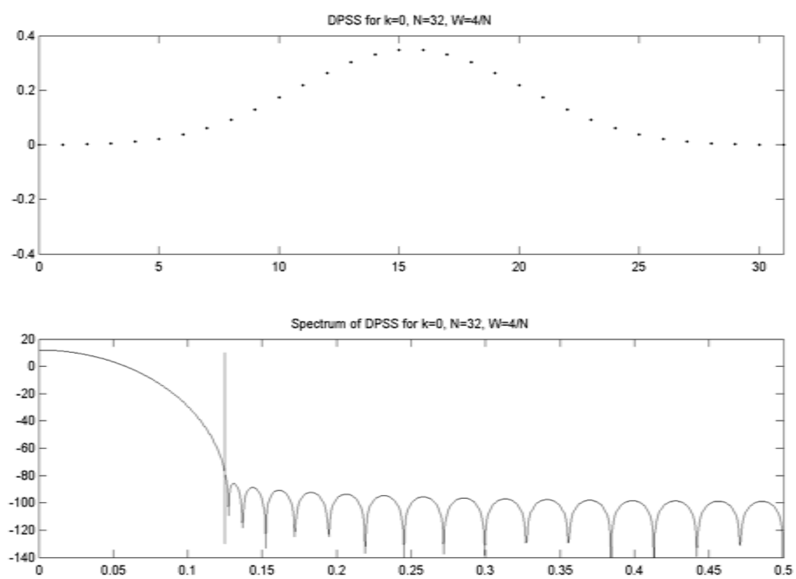
$$g_t = \frac{1}{\lambda_0(N,W)} \int_{-W}^{W} U_0(f;N,W) e^{i2\pi f[t-(N-1)/2]} df, t = 0,\pm1,\pm2$$

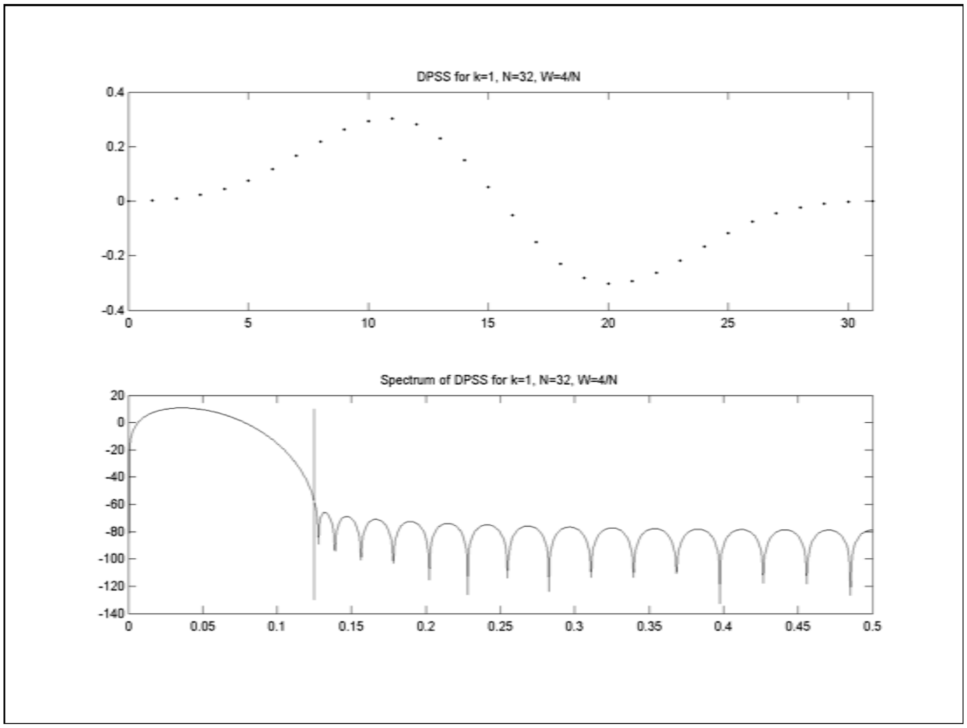
- The function U_k(.;N,W) is called the k^{th} order *discrete prolate spheroidal wave function* (dsswf) and the sequence {gt} is the zero-th order *discrete prolate spheroidal sequence* (dpss). The kth order dpss can be generated using the kth order dsswf.
- The matlab function dpss implements these Slepian sequences.
- PW discusses several methods for computing these functions. All are variations on solving an eigenvalue problem.

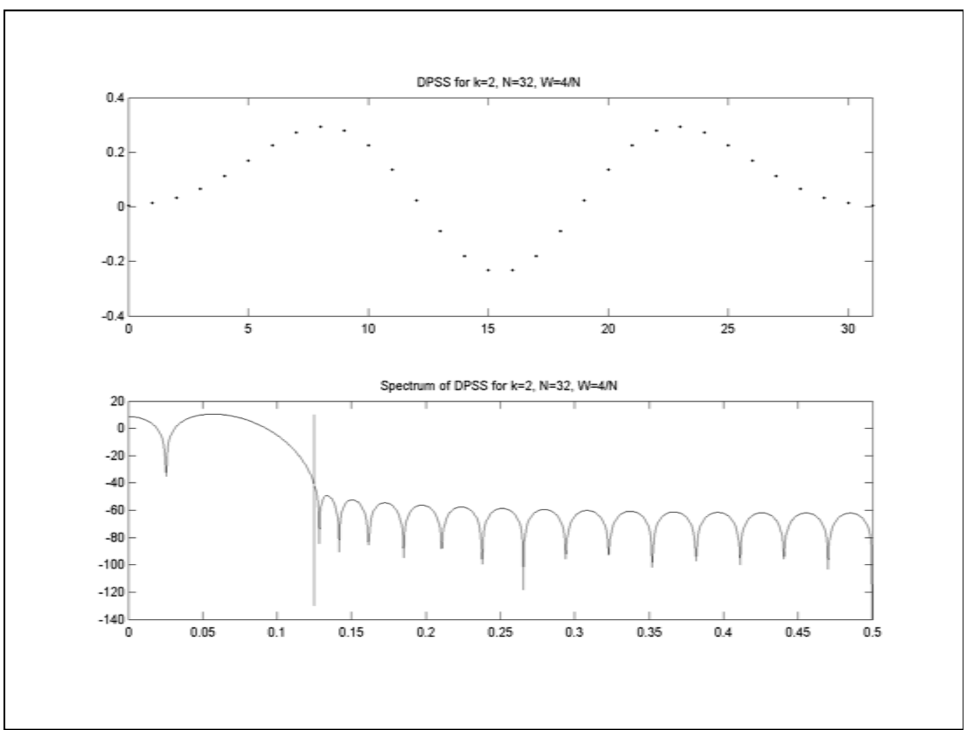
Examples

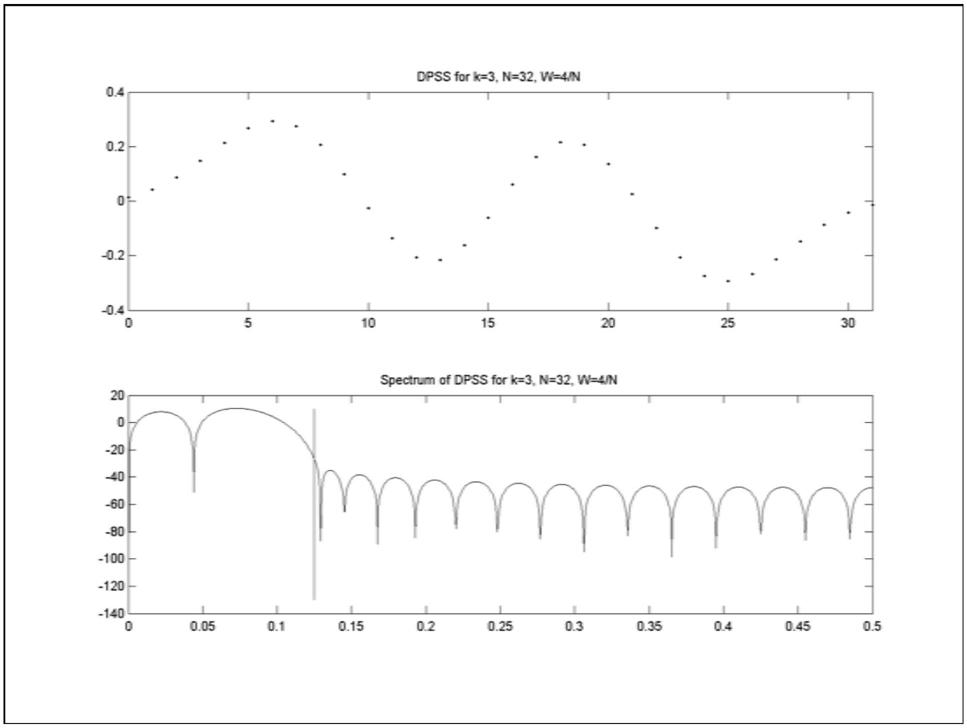
- The following figures show examples of dpss' s for values of N and W (N is number of data, W is bandwidth).
- The matlab code that goes with this lecture shows the method of calculation. Note: due to specific selections of frequencies in the spectra, the power does not drop completely to zero as it does in the text book figures.
- The figure show the first 4 order for N=32, W=4/N and N=99, W=4/N.
- The final figure shows the eigenvalues for both cases (2WN should be near unity which is these cases is 8)

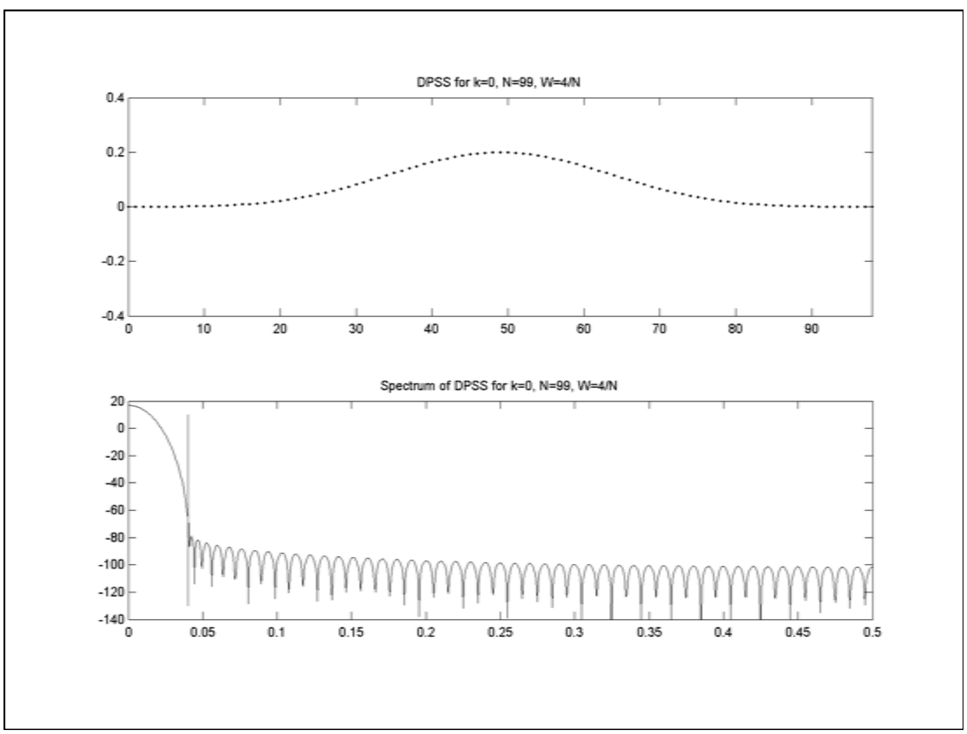
DPSS: N=32, k=0

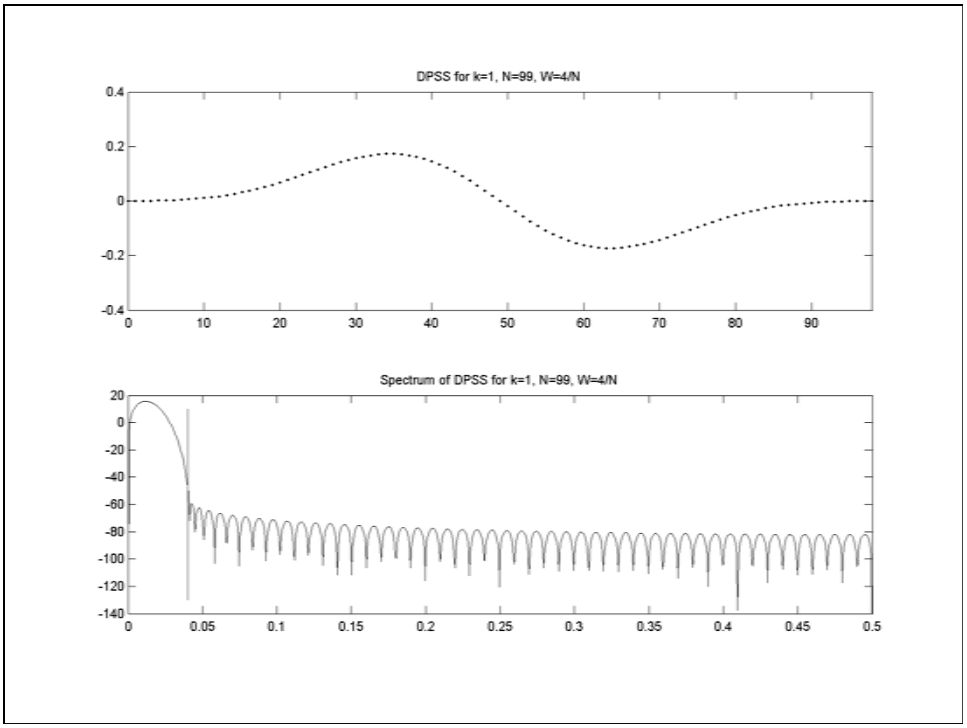


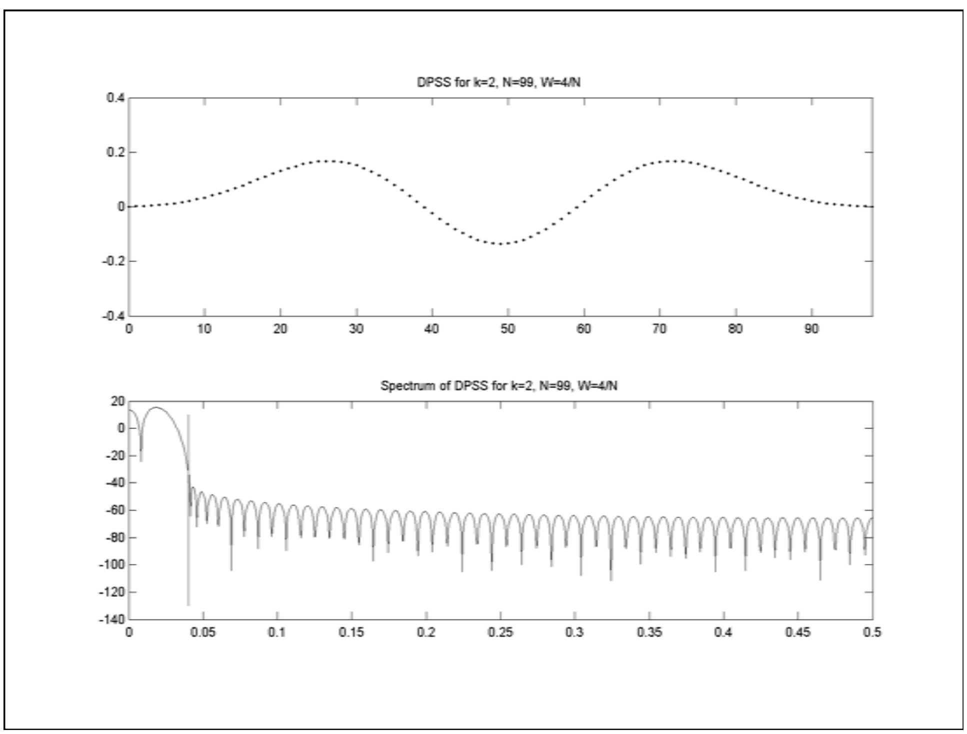

DPSS for k=1, N=32, W=4/N
Spectrum of DPSS for k=1, N=32, W=4/N

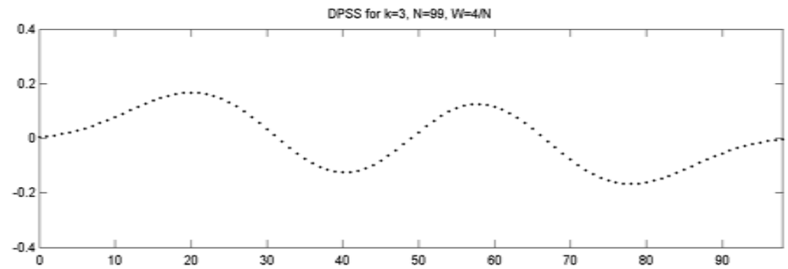

DPSS for k=2, N=32, W=4/N
Spectrum of DPSS for k=2, N=32, W=4/N

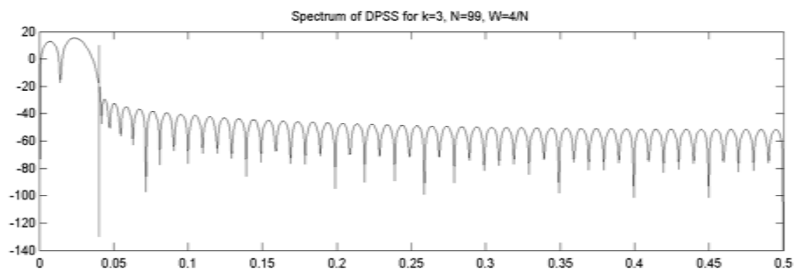

DPSS for k=3, N=32, W=4/N
Spectrum of DPSS for k=3, N=32, W=4/N


DPSS for k=0, N=99, W=4/N
Spectrum of DPSS for k=0, N=99, W=4/N

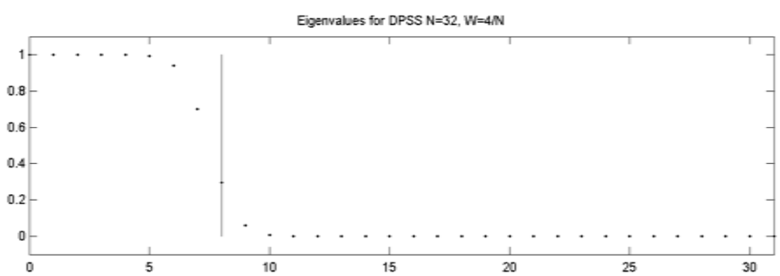

DPSS for k=1, N=99, W=4/N
0.4
0.2
0
-0.2
-0.4
0 10 20 30 40 50 60 70 80 90
Spectrum of DPSS for k=1, N=99, W=4/N
20
0
-20
-40
-60
-80
-100
-120
-140
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5

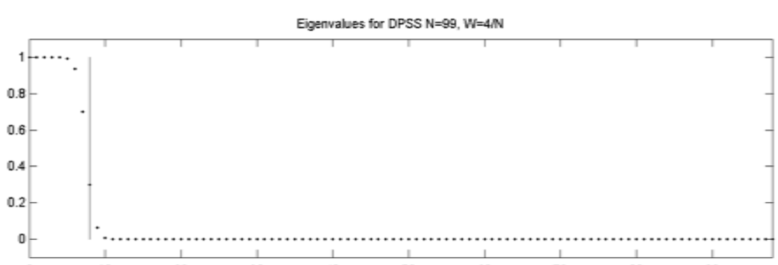

DPSS for k=2, N=99, W=4/N
0.4
0.2
0
-0.2
-0.4
0 10 20 30 40 50 60 70 80 90
Spectrum of DPSS for k=2, N=99, W=4/N
20
0
-20
-40
-60
-80
-100
-120
-140
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5





Eigenvalues for N=32 and 99





Properties of DPSS

- There are N non-zero eigenvalues that solve both the first and second concentration problems. The values are the same for both problems.
- The eigenvectors can be normalized so that they are orthonormal.
- The eigenvectors form a basis system in N-dimensional space that be used to expand any function through a linear combination of the basis functions. (Similar to Fourier expansion)
- The eigenvalues give the degree of concentration.
- Return to these functions in digital filters and multitaper spectral estimation.

Fourier Theory: Discrete time and Frequency

- The most common style of Fourier analysis involves discrete time and discrete frequency.
- With the appropriate choice of frequencies, the Fourier transforms can be implemented with the Fast Fourier Transform (FFT) algorithm which uses the "double angle" formulas for sine and cosine and thus avoids the (time-consuming) task of explicitly computing all the trig functions that appear in the calculation.

$$G_p(f) = \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi f t \Delta t}$$

For time limited $G_p(f;0,N-1) = \Delta t \sum_{t=0}^{N-1} g_t e^{-i2\pi f t \Delta t}$

Discrete FT

- Since we have a finite N number of data in the discrete case, we can define a discrete number of frequencies call the *Fourier frequencies* or *standard frequencies* given by $\boxed{f_n \equiv \frac{n}{N\Delta t} \text{ with } n = 0,1,\ldots,N-1}$
- Two "standards"

$$\boxed{\begin{array}{l} \Delta t = 1; \quad g_t = \frac{1}{N}\sum_{n=0}^{N-1} G_n e^{i2\pi nt/N} \text{ and } G_n = \sum_{t=0}^{N-1} g_t e^{-i2\pi nt/N} \\ \Delta t = 1/N; \quad g_t = \sum_{n=0}^{N-1} G_n e^{i2\pi nt/N} \text{ and } G_n = \frac{1}{N}\sum_{t=0}^{N-1} g_t e^{-i2\pi nt/N} \end{array}}$$

Notes on FFT definition

- In the previous expressions; if frequencies beyond n=0 to N-1 are computed, the resultant spectrum is periodic with period N; if times are computed beyond the n=0 to N-1 range, the time series is periodic as well. This behavior is often interpreted as the FFT being the Fourier Transform of an infinitely repeating sequence of the sample.
- Another interpretation is that all the g_t values outside the range 0-N are zero (The sums can then be extended to ±infinity. The infinite sequence would have the same values at the frequencies computed from the finite sample.

Relationship between G_n and $G_p(f)$

- Relationship between the true spectrum and the FFT can be found by noting that given the true spectrum we can compute the time series sample values and compute the FFT from these values

$$G_n = \Delta t \sum_{t=0}^{N-1} \left(\int_{-f_{(N)}}^{f_{(N)}} G_p(f) e^{i2\pi f t \Delta t} df \right) e^{-i2\pi n t / N}$$

$$= \Delta t \int_{-f_{(N)}}^{f_{(N)}} G_p(f) \sum_{t=0}^{N-1} e^{i2\pi t (f - f_n)} df \text{ where } f_n = n/(N\Delta t)$$

Relationship between G_n and $G_p(f)$

- If we look back to the lecture on Dirichlet's kernel, the equation can be rewritten as:

$$G_n = \int_{-f_{(N)}}^{f_{(N)}} G_p(f) P(f_n - f) df \equiv G_p * P(f_n)$$

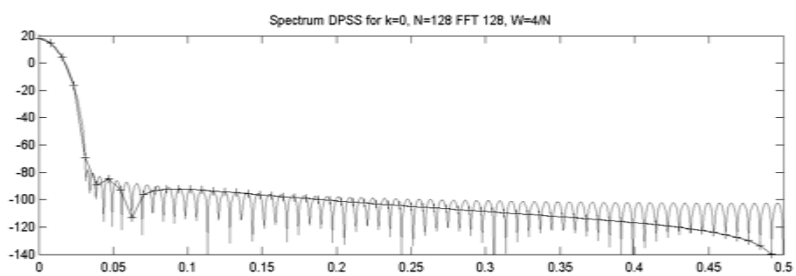
$$P(f_n) \equiv \Delta t e^{-i(N-1)\pi f \Delta t} N D_N (f \Delta t)$$

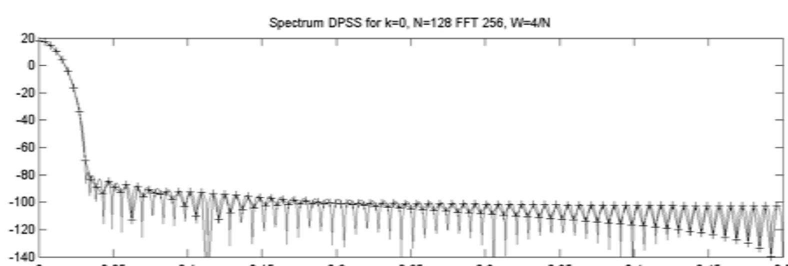
- We see P(.) acts as a blurring function since it is convolved with the actual spectrum $G_p(f)$. (Knowing the convolved result does not necessarily allow the generation of the original spectrum from the "smoothed one". Process is known as de-convolution.

Examples of DFFT

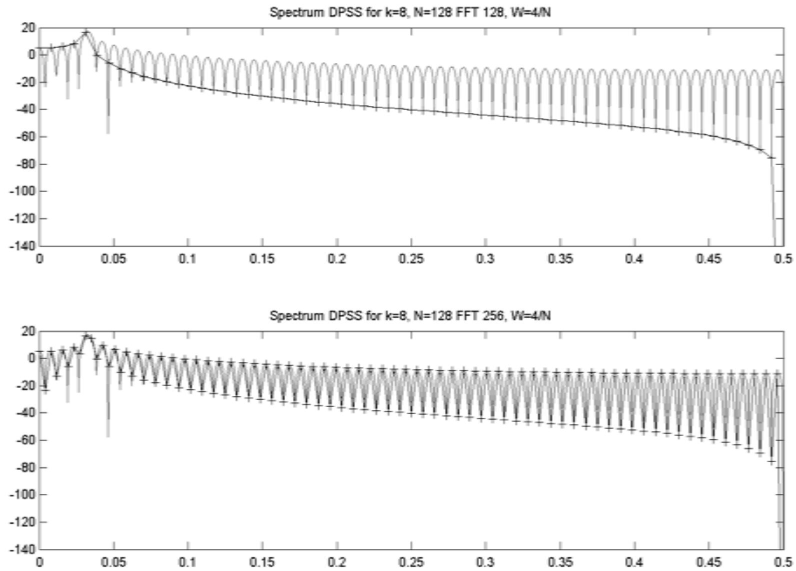
- For the previous cases of the DPSS functions the following examples show comparison of direct FFT calculation and the Matlab fft routine returns
- Two examples are shown for N=128, W=4/128, for k=0 and k=8 (k=8 is first eigenvalue < 0.5).
- Only the first N/2 values are shown from fft results. Since the input signal is real, the greater than N/2 values are symmetric about the mid-point (i.e., N=128 is N=-1; N=128 is N=-2).

FFT examples





FFT Example k=8



Material covered today

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