12.714 Computational Data Analysis

Alan Chave (alan@whoi.edu)
Thomas Herring (tah@mit.edu),
http://geoweb.mit.edu/~tah/12.714

Deterministic Spectral Analysis

- Today's class
  - Fourier Theory: Continuous time/Discrete frequency
  - Fourier Theory: Continuous time and frequency
  - Examples of transforms
  - Fourier transform theorems (from Bracewell, R, N., The Fourier Transform and its Applications, McGraw-Hill Book Company, New York, pp. 444, 1978)
  - Band-limited and time limited functions
  - Continuous/Continuous reciprocity relationships

Fourier Theory

- Examine in this class definition of various spectra for deterministic functions of time (again could be other sequential quantity)
- Rational:
  - A realization of a stochastic process is "deterministic" and so material here motivates the definition of spectrum for stationary process
  - Concept here (reciprocity, tapers) apply to deterministic and stochastic processes
  - Properties of deterministic functions appear in many stochastic processes.

Fourier Theory - Continuous time/ Discrete frequency

- Given that cos(2πnt/T) and sin(2πnt/T) define periodic functions of t with T>0, we can write a general periodic function as

$$\tilde{g}_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nt/T) + b_n \sin(2\pi nt/T)$$

with $\{a_n\}$ and $\{b_n\}$ (real or complex) constant such sum converges for all t.

- This expression can be written more compactly as

$$\tilde{g}_p(t) = \sum_{n=-\infty}^{\infty} G_n e^{i2\pi f_n t} \quad \text{: eqn (1) where } f_n = n/T \text{ and}$$

$$G_n \equiv \begin{cases} (a_n - ib_n)/2, & \text{for } n \geq 1 \\ (a_0/2), & \text{for } n = 0 \\ (a_n + ib_n)/2, & \text{for } n \leq -1 \end{cases}$$

Fourier Theory Expansion

- The use of negative frequencies is convenient mathematical trick that allow simplification of expressions and an easy addition for complex and real functions.
- If a and b are real then $G^*_{-n}=G_n$ and $|G_{-n}|=|G_n|$
- Any bounded, periodic function can be, in a certain sense, be represented by the expansion on the previous slide.
- This result can be shown by defining:

$$\boxed{g_{p,m}(t) = \sum_{n=-m}^{m} G_n e^{i2\pi f_n t} \text{ with } G_n = \frac{1}{T}\int_{-T/2}^{T/2} g_p(t) e^{-i2\pi f_n t} dt}$$

Fourier Theory: Expansion

- As m goes to infinity: $g_{p,m}$ will converge to g_p in mean square sense i.e., $\boxed{\lim_{m \to \infty} \int_{-T/2}^{T/2} \left| g_p(t) - g_{p,m}(t) \right|^2 dt = 0}$
- This type of equality is denoted with a ms over the equals sign.
- Relationship derived with orthogonality relation

$$\boxed{\int_{-T/2}^{T/2} e^{i2\pi(f_n - f_m)t} dt = \begin{cases} 0, & m \neq n \\ T, & m = n \end{cases}}$$

- Eqn (1) is the *Fourier series representation*, G_n in n^{th} *Fourier coefficient*.

Energy in time series

- Energy in time series over period [-T/2,T/2] is the integral over the interval of $g_p(t)^2$. Using orthogonality we can show

$$\int_{-T/2}^{T/2} \left|g_p(t)\right|^2 dt = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_n G_m^* \int_{-T/2}^{T/2} e^{i2\pi(f_n - f_m)t} dt = T \sum_{n=-\infty}^{\infty} \left|G_n\right|^2$$

- This is *Parseval's Theorem* (or *Rayleigh's Theorem*) for Fourier series.
- Since the function is periodic, there is infinite energy over infinite time and so the concept of *power* is introduced: Energy per unit time. This is the above equation divided by T
- While the energy is infinite, over infinite time, the power is finite.
- The *discrete power spectrum* is defined to be $S_n=|G_n|^2$

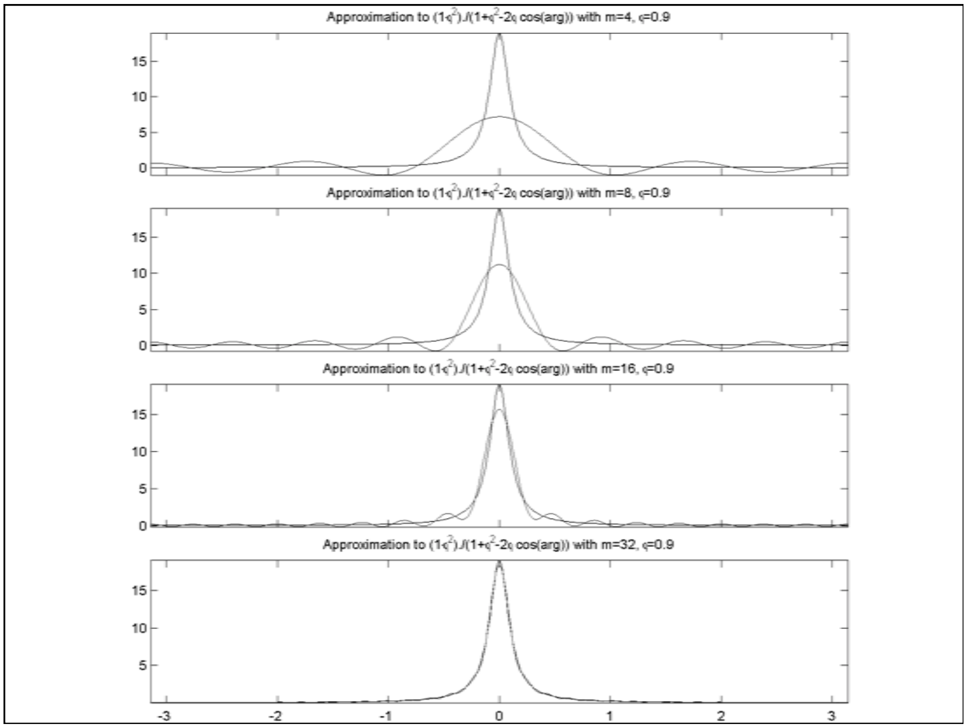
Discrete Power Spectrum

- The original signal can not be recovered from the power spectrum but can be recovered for the Fourier coefficients themselves
- The following example demonstrates this.

$$g_p(t) = \frac{1-\varphi^2}{1+\varphi^2 - 2\varphi\cos(t)}$$

$$g_{p,m}(t) = \sum_{n=-m}^{m} G_n e^{i2\pi f_n t} = 1 + 2\sum_{n=1}^{m} \varphi^n \cos(nt)$$

$$S_n = \varphi^{2|n|} \text{ is the discrete power spectrum}$$



Discrete Fourier series

- One question that arises in the previous example is: If we are going to truncate the Fourier series to order m, are the G_n that we determine from the Fourier series the best choice or would different coefficients be better?
- Question addressed p 60 of PW and when the energy of the difference between $g_p(t)$ and the approximation to it is used, the use of the Fourier coefficients minimizes the energy of the difference. This is the least squares fit.

Fourier Theory: Continuous Time and Frequency

- Suppose we have a non-periodic function. We can not expand this function in Fourier series (i.e., discrete frequencies).
- However, if we take a section of the function over interval T, and the function is square integrable over the interval, and replicate this piece so that it is periodic, we can expand in Fourier Series as before.

$$g_T(t) \overset{ms}{=} \sum_{n=-\infty}^{\infty} G_{n,T} e^{i2\pi f_n t} \text{ where T is [-T/2,T/2]}$$

$$\mathrm{G}_{\mathrm{n,T}} = \frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{-i2\pi f_n t} dt$$

Fourier Theory: Continuous time and frequency

- Given the expansion for interval -T/2 to T/2, we have

$$g(t) \equiv g_T(t) \overset{ms}{=} \sum_{n=-\infty}^{\infty} \left(\int_{-T/2}^{T/2} g(t) e^{-i2\pi f_n t} dt \right) e^{i2\pi f_n t} \Delta f; \quad \Delta f = 1/T$$

As $\mathrm{T} \to \infty$, $\Delta f \to 0$ and summations become integrals

$$g(t) \overset{ms}{=} \int_{-\infty}^{\infty} G(f) e^{i2\pi f t} df, \quad G(f) \equiv \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t} dt$$

- The equations above are the *Fourier integral representation* and G(f) is the *Fourier Transform* of g(t).
- (In the literature you will see different normalizations of the Fourier and inverse Fourier transforms)

Fourier Transform

- The function pair g(t) ←→G(f) are Fourier transform pairs. In general, G(f) is complex and can written as $G(f) = |G(f)|e^{i\theta(f)}$ where $\theta(f)$ is a phase as a function of frequency.
- |G(f)| is often referred to as the *amplitude spectrum*
- The energy in the function is related to G through

$$\boxed{\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df}$$

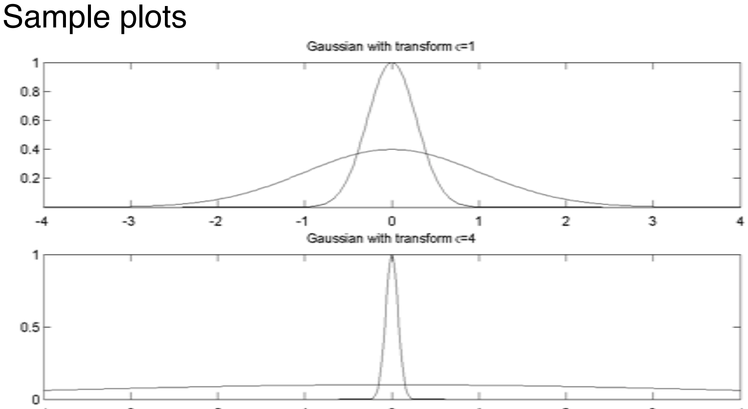
- $|G(f)|^2$ $(=G^*(f)G(f))$ is called the energy spectral density function.

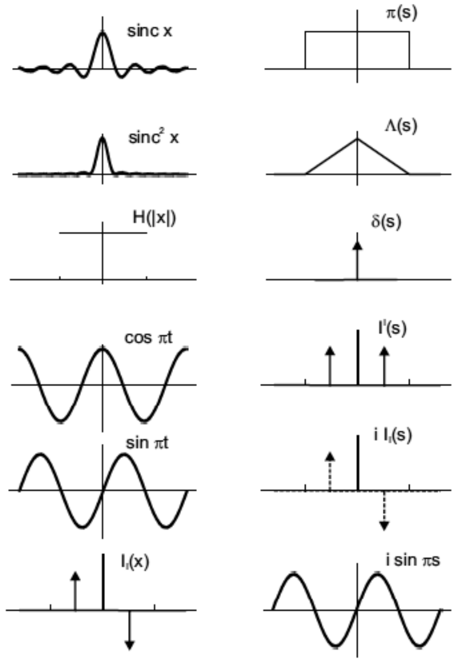
Example

- Gaussian Probability density function:

$$\boxed{g_\sigma(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)} \quad G_\sigma(f) = e^{-2\pi^2 f^2 \sigma^2}}$$

- Sample plots





Examples of common transforms

Bracewell p. 386-398 has a "pictorial" set of transforms

Note: the i' s on some transform. Real functions often have imaginary transforms.

Fourier Transform Theorems

- The following is set of basic theorems commonly encountered in working with Fourier transforms.
- Here we use x and s as domain variables, f(x) and F(s) are the transform pair
- *Similarity theorem*: $\boxed{\int_{-\infty}^{\infty} f(ax)e^{-i2\pi xs}dx = \frac{1}{a}F\left(\frac{s}{a}\right)}$
- *Addition theorem*: $\boxed{\int_{-\infty}^{\infty} [f(x)+g(x)]e^{-i2\pi xs}dx = F(s)+G(s)}$
- *Shift theorem*: $\boxed{\int_{-\infty}^{\infty} f(x-a)e^{-i2\pi xs}dx = e^{-i2\pi as}F(s)}$

Fourier Transform Theorems

- *Modulation theorem*:

$$\boxed{\int_{-\infty}^{\infty} f(x)\cos(\omega x)e^{-i2\pi xs}dx = \frac{1}{2}F(s-\omega/(2\pi)) + \frac{1}{2}F(s+\omega/(2\pi))}$$

- *Convolution theorem*:

$$h(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$
$$H(s) = F(s)G(s)$$
$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x')g(x-x')dx'\right]e^{-i2\pi xs}dx = \int_{-\infty}^{\infty} f(x')e^{-i2\pi x's}G(s)dx' = F(s)G(s)$$

Fourier Transform Theorems

- *Rayleigh's Theorem*: $\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds}$
- *Power Theorem*: $\boxed{\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} F(s)G^*(s)ds}$
- *Autocovariance theorem*: Fourier transform of autocovariance function is transform squared.

$$\boxed{\int_{-\infty}^{\infty} |F(s)|^2 e^{i2\pi xs}ds = \int_{-\infty}^{\infty} f^*(u)f(u+x)du}$$

Fourier Transform Theorems

- *Derivative theorem*: $\boxed{\int_{-\infty}^{\infty} f'(x) e^{-i2\pi xs} dx = i2\pi s F(s)}$

  where f' (x) is the derivative of the function with respect to x. (Theorem proved with shift theorem).
- *Derivative of a convolution*: Derivative of a convolution is the convolution with the derivative of either function. (Shown with theorem above since one derivative to multiply transform by i2πs, then inverse will generate derivative.

Band-Limited and Time-limited functions

- Most signals have a high-frequency cutoff (mainly due to high frequency losses). In some cases, there are low frequency cut offs as well (e.g., seismic recorders). If there is no energy above a frequency W, the signal is *band-limited* in the band [-W,W].
- These signals are smooth due to finite higher derivatives (a Taylor series expansion exists for every point).
- A *time-limited* signal is one that is zero for all ItI>T/2. (Seismic signals after earthquakes are time-limited).
- The only time-limited and band-limited signal is zero. (Because a Taylor Series expansion can be made anywhere, choosing a time when the signal is zero (and all derivatives are zero) shows the whole function must be zero).

Reciprocity Relationships: Continuous/Continuous

- Relationship between time and frequency domains leading to fundamental uncertainty relationships
- *Similarity theorem*: if g(.) and G(.) are a Fourier transform pair then

$$|a|^{1/2} g(at) \Leftrightarrow \frac{1}{|a|^{1/2}} G(f/a) \text{ are Fourier Transform Pairs}$$

- Thus if one domain expands horizontally and vertically the other contracts in the corresponding directions
- This form is often shown with the lal scaling on just one side of the equation.

Equivalent Width

- For real functions that are positive, peaked near zero we can define an equivalent width as the width needed for a box of the height at g(0) so that area of the box is the same as the integral of the function.
- Because of reciprocity we can do this is both Fourier domains and the widths are inversely related i.e.,

$$\text{width}\{g(.)\} = \int_{-\infty}^{\infty} g(t)dt \,/\, g(0) = \frac{G(0)}{\int_{-\infty}^{\infty} G(f)df} = 1 \big/ \text{width}\{G(.)\}$$

- These concepts used in defining bandwidth. Note: compact signals in one domain (e.g. short pulse) are wide in the other domain.
- Example: Internet bandwidth, more bandwidth shorter time to transmit bit thus faster speed.

Fundamental Uncertainty Relationship

- This is a version of the Heisenberg's uncertainty principle. Instead of using area under the function to define width we use the second moment or variance.

$$\sigma_{\tilde{g}}^2 = \int_{-\infty}^{\infty} (t-\mu_{\tilde{g}})^2 \tilde{g}(t)dt, \text{ where } \mu_{\tilde{g}} = \int_{-\infty}^{\infty} t\tilde{g}(t)dt$$

$$\text{and } \tilde{g}(t) = g(t) / \int_{-\infty}^{\infty} \tilde{g}(t)dt \quad \text{with} \quad 0 < \int_{-\infty}^{\infty} \tilde{g}(t)dt = C < \infty$$

$$\text{width}_v\{g(.)\} \equiv 2\sqrt{3}\sigma_{\tilde{g}}$$

- In this form, we are treating g(.) as a probability density function
- With some derivation the Heisenberg uncertainty principle can be shown: $\boxed{\sigma_g^2\sigma_G^2 \geq 1/16\pi^2}$

Summary of today's class

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